

Stochastic differential equations related to random matrix theory

By

Hirofumi OSADA* and Hideki TANEMURA**

Abstract

In this note we review recent results on existence and uniqueness of solutions of infinite-dimensional stochastic differential equations describing interacting Brownian motions on \mathbb{R}^d .

§ 1. Introduction

Let $\mathbf{X}^N(t) = (X_j^N(t))_{j=1}^N$ be a solution of the stochastic differential equation (SDE)

$$(1.1) \quad dX_j^N(t) = dB_j(t) + \frac{\beta}{2} \sum_{k=1, k \neq j}^N \frac{dt}{X_j^N(t) - X_k^N(t)}$$

or the SDE with Ornstein-Uhlenbeck's type drifts

$$(1.2) \quad dX_j^N(t) = dB_j(t) - \frac{\beta}{4N} X_j^N(t) dt + \frac{\beta}{2} \sum_{k=1, k \neq j}^N \frac{dt}{X_j^N(t) - X_k^N(t)},$$

where $B_j(t), j = 1, 2, \dots, N$ are independent one-dimensional Brownian motions. These are called Dyson's Brownian motion models with parameters $\beta > 0$ [4]. They were introduced to understand the statistics of eigenvalues of random matrix ensembles as

Received April 20, 201x. Revised September 11, 201x.

2010 Mathematics Subject Classification(s): 15B52, 30C15, 47D07, 60G55, 82C22

Key Words: Random matrices, Interacting particle systems, Stochastic differential equations:

Supported by a Grant-in-Aid for Scientific Research (KIBAN-A, No.24244010) and a Grant-in-Aid for Scientific Research (KIBAN-C, No.15K04916)

*Faculty of Mathematics, Kyushu University, Fukuoka 819-0395 Japan.

e-mail: osada@math.kyushu.ac.jp

**Department of Mathematics and Informatics, Faculty of Science, Chiba University, 1-33 Yayoi-cho, Inage-ku, Chiba 263-8522, Japan.

e-mail: tanemura@math.s.chiba-u.ac.jp

distributions of particle positions in one-dimensional Coulomb gas systems with log-potential.

The solution of (1.2) is a natural reversible stochastic dynamics with respect to $\tilde{\mu}_{\text{bulk},\beta}^N$:

$$(1.3) \quad \tilde{\mu}_{\text{bulk},\beta}^N(d\mathbf{x}_N) = \frac{1}{Z} h_N(\mathbf{x}_N)^\beta e^{-\frac{\beta}{4N}|\mathbf{x}_N|^2} d\mathbf{x}_N,$$

where $d\mathbf{x}_N = dx_1 dx_2 \cdots dx_N$, $\mathbf{x}_N = (x_i) \in \mathbb{R}^N$, and

$$h_N(\mathbf{x}_N) = \prod_{i < j}^N |x_i - x_j|.$$

Throughout, Z denotes a normalizing constant. Gaussian ensembles are called Gaussian orthogonal/unitary/symplectic ensembles (GOE/GUE/GSE) according to their invariance under conjugation by orthogonal/unitary/symplectic groups, which correspond to the inverse temperatures $\beta = 1, 2$ and 4 , respectively [9, 2]. It is natural to believe that the N -limit of the process $\mathbf{X}^N(t)$ solves the infinite-dimensional stochastic differential equation (ISDE)

$$(1.4) \quad dX_j(t) = dB_j(t) + \frac{\beta}{2} \lim_{r \rightarrow \infty} \sum_{\substack{k=1, k \neq j \\ |X_k(t)| < r}}^{\infty} \frac{dt}{X_j(t) - X_k(t)}.$$

The result was not proved rigorously until a few years ago when it was shown for $\beta = 2$ in [17], for $\beta = 1, 2, 4$ in [8], and for $\beta \geq 1$ in [23].

Set $Y_j^N(t) = N^{1/6}(X_j^N(t) - 2\sqrt{N})$, $j = 1, 2, \dots, N$ for the solution \mathbf{X}^N of (1.2). It has also been shown that the N -limit of the process $\mathbf{Y}^N(t)$ solves the ISDE

$$(1.5) \quad dY_j(t) = dB_j(t) + \frac{\beta}{2} \lim_{r \rightarrow \infty} \left\{ \sum_{\substack{k=1, k \neq j \\ |Y_k(t)| < r}}^{\infty} \frac{1}{Y_j(t) - Y_k(t)} - \int_{-r}^r \frac{\widehat{\rho}(x) dx}{-x} \right\} dt,$$

with $\widehat{\rho}(x) = \pi^{-1} \sqrt{-x} \mathbf{1}(x < 0)$, for $\beta = 2$ [17] and for $\beta = 1, 2, 4$ [8].

One of the key parts of proving the above results is the existence and uniqueness of solutions of an ISDE of the form

$$(1.6) \quad dX_j(t) = dB_j(t) - \frac{1}{2} \nabla \Phi(X_j(t)) dt - \frac{1}{2} \sum_{k=1, k \neq j}^{\infty} \nabla \Psi(X_j(t), X_k(t)) dt$$

with free potential Φ and interaction (pair) potential Ψ . In ISDEs (1.4) and (1.5), Ψ is given by the log pair potential $-\beta \log |x - y|$. The present note is a short summary of results on existence and uniqueness of solutions for ISDE (1.6).

§ 2. Quasi-Gibbs measure

Let S be a closed set in \mathbb{R}^d such that $0 \in S$ and $\overline{S^{\text{int}}} = S$, where S^{int} denotes the interior of S . The configuration space \mathfrak{M} of unlabelled particles is given by

$$(2.1) \quad \mathfrak{M} = \left\{ \xi : \xi \text{ is a nonnegative integer valued Radon measure in } S \right\} \\ = \left\{ \xi(\cdot) = \sum_{j \in \mathbb{I}} \delta_{x_j}(\cdot) : \sharp\{j \in \mathbb{I} : x_j \in K\} < \infty, \text{ for any } K \text{ compact} \right\},$$

where \mathbb{I} is a countable set and δ_a is the Dirac measure at $a \in S$. Thus \mathfrak{M} is a Polish space with the vague topology. We also introduce a subset $\mathfrak{M}_{\text{s.i.}}$ of \mathfrak{M} :

$$(2.2) \quad \mathfrak{M}_{\text{s.i.}} = \{ \xi \in \mathfrak{M} : \xi(\{x\}) \leq 1 \text{ for all } x \in S, \xi(S) = \infty \},$$

that is, the set of configurations of an infinite number of particles without collisions. For Borel measurable functions $\Phi : S \rightarrow \mathbb{R} \cup \{\infty\}$ and $\Psi : S \times S \rightarrow \mathbb{R} \cup \{\infty\}$ and a given increasing sequence $\{b_r\}$ of \mathbb{N} , we introduce the Hamiltonian

$$(2.3) \quad H_r(\xi) = H_r^{\Phi, \Psi}(\xi) = \sum_{x_j \in S_r} \Phi(x_j) + \sum_{x_j, x_k \in S_r, j < k} \Psi(x_j, x_k), \quad \xi = \sum_{j \in \mathbb{I}} \delta_{x_j},$$

where $S_r = \{x \in S : |x| < b_r\}$. We call Φ a free potential, and call Ψ an interaction potential. Let Λ_r^m be the restriction of a Poisson random measure with intensity measure dx on $\mathfrak{M}_r^m = \{\xi \in \mathfrak{M} : \xi(S_r) = m\}$. We define maps $\pi_r, \pi_r^c : \mathfrak{M} \rightarrow \mathfrak{M}$ such that $\pi_r(\xi) = \xi(\cdot \cap S_r)$ and $\pi_r^c(\xi) = \xi(\cdot \cap S_r^c)$. For two measures ν_1, ν_2 on a measurable space (Ω, \mathcal{F}) we write $\nu_1 \leq \nu_2$ if $\nu_1(A) \leq \nu_2(A)$ for any $A \in \mathcal{F}$. We can now state the definition of a *quasi-Gibbs measure* [13, 14].

Definition 2.1. A probability measure μ on \mathfrak{M} is said to be a (Φ, Ψ) -quasi Gibbs measure if its regular conditional probabilities

$$\mu_{r, \xi}^m(d\zeta) = \mu(d\zeta | \pi_r^c(\zeta) = \pi_r^c(\xi), \zeta(S_r) = m), \quad r, m \in \mathbb{N},$$

satisfy that, for μ -a.s. ξ ,

$$c^{-1} e^{-H_r(\eta)} \Lambda_r^m(\pi_{S_r} \in d\eta) \leq \mu_{r, \xi}^m(\pi_{S_r} \in d\eta) \leq c e^{-H_r(\eta)} \Lambda_r^m(\pi_{S_r} \in d\eta).$$

Here, $c = c(r, m, \xi)$ is a positive constant depending on r, m , and ξ .

It is readily seen that the quasi-Gibbs property is a generalized notion of the canonical Gibbs property. If μ is a (Φ, Ψ) -quasi Gibbs measure, then μ is also a $(\Phi + \Phi_{\text{loc.bdd}}, \Psi)$ -quasi Gibbs measure for any locally bounded measurable function $\Phi_{\text{loc.bdd}}$. In this sense, the notion of “quasi-Gibbs” seems to be robust. Information about the free potential of μ is determined from its *logarithmic derivative* [12].

A function f on \mathfrak{M} is called a polynomial function if

$$(2.4) \quad f(\xi) = Q(\langle \phi_1, \xi \rangle, \langle \phi_2, \xi \rangle, \dots, \langle \phi_\ell, \xi \rangle)$$

with $\phi_k \in C_c^\infty(S)$ and a polynomial function Q on \mathbb{R}^ℓ , where $\langle \phi, \xi \rangle = \int_S \phi(x) \xi(dx)$ and $C_c^\infty(S)$ is the set of smooth functions with compact support. We denote by \mathcal{P} the set of all polynomial functions on \mathfrak{M} .

Definition 2.2. We call $\mathbf{d}^\mu \in L_{loc}^1(S \times \mathfrak{M}, \mu^{[1]})$ the logarithmic derivative of μ if

$$\int_{S \times \mathfrak{M}} \mathbf{d}^\mu(x, \eta) f(x, \eta) d\mu^{[1]}(x, \eta) = - \int_{S \times \mathfrak{M}} \nabla_x f(x, \eta) d\mu^{[1]}(x, \eta)$$

is satisfied for $f \in C_c^\infty(S) \otimes \mathcal{P}$. Here $\mu^{[k]}$ is the Campbell measure of μ

$$\mu^{[k]}(A \times B) = \int_A \mu_{\mathbf{x}}(B) \rho^k(\mathbf{x}) d\mathbf{x}, \quad A \in \mathcal{B}(S^k), B \in \mathcal{B}(\mathfrak{M}),$$

$\mu_{\mathbf{x}}$ is the reduced Palm measure conditioned at $\mathbf{x} \in S^k$

$$(2.5) \quad \mu_{\mathbf{x}} = \mu \left(\cdot - \sum_{j=1}^k \delta_{x_j} \middle| \xi(x_j) \geq 1 \text{ for } j = 1, 2, \dots, k \right),$$

and ρ^k is the k -correlation function for $k \in \mathbb{N}$.

Quasi-Gibbs measures inherit the following property from canonical Gibbs measures [19, Lemma 11.2]. Let $\mathcal{T}(\mathfrak{M})$ be the tail σ -field

$$\mathcal{T}(\mathfrak{M}) = \bigcap_{r=1}^{\infty} \sigma(\pi_r^c)$$

and let μ_{Tail}^ξ be the regular conditional probability defined as

$$(2.6) \quad \mu_{\text{Tail}}^\xi = \mu(\cdot | \mathcal{T}(\mathfrak{M}))(\xi).$$

Then the following decomposition holds:

$$(2.7) \quad \mu(\cdot) = \int_{\mathfrak{M}} \mu_{\text{Tail}}^\xi(\cdot) \mu(d\xi).$$

Furthermore, there exists a subset \mathfrak{M}_0 of \mathfrak{M} satisfying $\mu(\mathfrak{M}_0) = 1$ and, for all $\xi, \eta \in \mathfrak{M}_0$:

$$(2.8) \quad \mu_{\text{Tail}}^\xi(A) \in \{0, 1\} \quad \text{for all } A \in \mathcal{T}(\mathfrak{M}),$$

$$(2.9) \quad \mu_{\text{Tail}}^\xi(\{\zeta \in \mathfrak{M} : \mu_{\text{Tail}}^\xi = \mu_{\text{Tail}}^\zeta\}) = 1,$$

$$(2.10) \quad \mu_{\text{Tail}}^\xi \text{ and } \mu_{\text{Tail}}^\eta \text{ are mutually singular on } \mathcal{T}(\mathfrak{M}) \text{ if } \mu_{\text{Tail}}^\xi \neq \mu_{\text{Tail}}^\eta.$$

§ 3. General theory of solutions of ISDEs

A polynomial function f on \mathfrak{M} is a *local* function, that is, a function satisfying $f(\xi) = f(\pi_r(\xi))$ for some $r \in \mathbb{N}$. When $\xi \in \mathfrak{M}_r^m$, $m \in \mathbb{N} \cup \{0\}$ and $\pi_r(\xi)$ is represented by $\sum_{j=1}^m \delta_{x_j}$, we can regard $f(\xi) = f(\sum_{j=1}^m \delta_{x_j})$ as a permutation invariant smooth function on S_r^m . For $f, g \in \mathcal{P}$, define

$$\mathbb{D}(f, g)(\xi) = \frac{1}{2} \sum_{j=1}^{\infty} \nabla_{x_j} f(\xi) \cdot \nabla_{x_j} g(\xi).$$

For a probability μ on \mathfrak{M} , we denote by $L^2(\mathfrak{M}, \mu)$ the space of square integrable functions on \mathfrak{M} with the inner product $\langle \cdot, \cdot \rangle_\mu$ and the norm $\| \cdot \|_{L^2(\mathfrak{M}, \mu)}$. We consider the bilinear form $(\mathcal{E}^\mu, \mathcal{P}^\mu)$ on $L^2(\mathfrak{M}, \mu)$ defined by

$$(3.1) \quad \mathcal{E}^\mu(f, g) = \int_{\mathfrak{M}} \mathbb{D}(f, g) d\mu, \quad \mathcal{P}^\mu = \{f \in \mathcal{P} : \|f\|_1^2 < \infty\},$$

where $\|f\|_1^2 \equiv \mathcal{E}^\mu(f, f) + \|f\|_{L^2(\mathfrak{M}, \mu)}^2$.

We make the following assumptions

(A.0) μ has a locally bounded n -correlation function ρ^n for each $n \in \mathbb{N}$.

(A.1) There exist upper semi-continuous functions $\Phi_0 : S \rightarrow \mathbb{R} \cup \{\infty\}$ and $\Psi_0 : S \times S \rightarrow \mathbb{R} \cup \{\infty\}$ that are locally bounded from below, and $c > 0$ such that

$$c^{-1}\Phi_0(x) \leq \Phi(x) \leq c\Phi_0(x), \quad c^{-1}\Psi_0(x, y) \leq \Psi(x, y) \leq c\Psi_0(x, y).$$

(A.2) There exists a $T > 0$ such that for each $R > 0$

$$\liminf_{r \rightarrow \infty} \text{Erf} \left(\frac{r}{(r+R)T} \right) \int_{|x| \leq r+R} \rho^1(x) dx = 0,$$

where $\text{Erf}(t) = (2\pi)^{-1/2} \int_t^\infty e^{-x^2/2} dx$.

Note that $\mathcal{P}^\mu = \mathcal{P}$ and $(\mathcal{E}^\mu, \mathcal{P}^\mu) = (\mathcal{E}, \mathcal{P})$ under condition (A.0).

Theorem 3.1 ([12, 13, 14, 11, 16]). *Suppose that μ is a (Φ, Ψ) -quasi Gibbs measure satisfying (A.0) and (A.1). Then*

(i) $(\mathcal{E}, \mathcal{P})$ is closable and its closure $(\mathcal{E}^\mu, \mathcal{D}^\mu)$ is a quasi regular Dirichlet form and there exists the diffusion process $(\Xi(t), P_\mu^\xi)$ associated with $(\mathcal{E}^\mu, \mathcal{D}^\mu)$.

(ii) Furthermore, assume conditions (A.2) and (A.3):

(A.3) $\text{Cap}^\mu((\mathfrak{M}_{s,i})^c) = 0$ and $\text{Cap}^\mu(\xi(\partial S) \geq 1) = 0$,

where Cap^μ is the capacity of the Dirichlet form. If there exists a logarithmic derivative \mathbf{d}^μ , then there exists $\tilde{\mathfrak{M}} \subset \mathfrak{M}$ such that $\mu(\tilde{\mathfrak{M}}) = 1$, and for any $\xi = \sum_{j \in \mathbb{N}} \delta_{x_j} \in \tilde{\mathfrak{M}}$, there exists an $S^\mathbb{N}$ -valued continuous process $\mathbf{X}(t) = (X_j(t))_{j=1}^\infty$ satisfying $\mathbf{X}(0) = \mathbf{x} = (x_j)_{j=1}^\infty$ and

$$dX_j(t) = dB_j(t) + \frac{1}{2} \mathbf{d}^\mu \left(X_j(t), \sum_{k:k \neq j} \delta_{X_k(t)} \right) dt, \quad j \in \mathbb{N}.$$

Let \mathfrak{l} be a label map from $\mathfrak{M}_{s.i.}$ to $S^{\mathbb{N}}$, that is, for each $\xi \in \mathfrak{M}_{s.i.}$, $\mathfrak{l}(\xi) = (\mathfrak{l}(\xi)_j)_{j=1}^{\infty} \in S^{\mathbb{N}}$ satisfies $\xi = \sum_{j=1}^{\infty} \delta_{\mathfrak{l}(\xi)_j}$. The map \mathfrak{l} can be lifted to the map from $C([0, \infty), \mathfrak{M}_{s.i.})$ to $C([0, \infty), S^{\mathbb{N}})$. For $\Xi \in C([0, \infty), \mathfrak{M}_{s.i.})$ we put

$$\Xi^{\diamond m}(t) = \sum_{j=m+1}^{\infty} \delta_{X_j(t)}$$

for each $m \in \mathbb{N}$, where $(X_j)_{j=1}^{\infty} = \mathfrak{l}(\Xi) \in C([0, \infty), S^{\mathbb{N}})$. We make the following assumption.

(A4) There exists a subset $\mathfrak{M}_{\text{SDE}}$ of $\mathfrak{M}_{s.i.}$ such that

$$P_{\mu}^{\xi}(\Xi(t) \in \mathfrak{M}_{\text{SDE}}) = 1 \quad \text{for any } \xi \in \mathfrak{M}_{\text{SDE}},$$

and for each $\Xi \in C([0, \infty), \mathfrak{M}_{\text{SDE}})$ and each $m \in \mathbb{N}$,

$$(3.2) \quad dY_j^{(m)}(t) = dB_j(t) - \frac{1}{2} \nabla \Phi(Y_j^{(m)}(t)) dt - \frac{1}{2} \sum_{k=1, k \neq j}^m \nabla \Psi(Y_j^{(m)}(t), Y_k^{(m)}(t)) dt$$

$$- \frac{1}{2} \int_{\mathfrak{M}} \nabla \Psi(Y_j^{(m)}(t), X(t)) \Xi^{\diamond m}(dX) dt, \quad 1 \leq j \leq m,$$

$$(3.3) \quad Y_j^{(m)}(0) = \mathfrak{l}(\Xi(0))_j, \quad 1 \leq j \leq m,$$

has a unique strong solution $\mathbf{Y}^{(m)} = (Y_1^{(m)}, Y_2^{(m)}, \dots, Y_m^{(m)})$.

We also make the following assumptions about the probability measure μ

(A5) For each $r, T \in \mathbb{N}$, there exists a positive constant c such that

$$\int_S \text{Erf} \left(\frac{|x| - r}{\sqrt{cT}} \right) \rho^1(x) dx < \infty.$$

(A6) The tail σ -field $\mathcal{T}(\mathfrak{M})$ is μ -trivial, that is, $\mu(A) \in \{0, 1\}$ for $A \in \mathcal{T}(\mathfrak{M})$.

Definition 3.2. Let μ be a probability measure on \mathfrak{M} and let $\Xi(t)$ be an \mathfrak{M} -valued process. We say that $\Xi(t)$ satisfies the μ -absolute continuity condition if $\mu \circ \Xi(t)^{-1}$ is absolutely continuous with respect to μ for $\forall t > 0$. We say that an $S^{\mathbb{N}}$ -valued process $\mathbf{X}(t)$ satisfies the μ -absolute continuity condition if $\mathbf{u}(\mathbf{X}(t))$ satisfies the μ -absolute continuity condition, where \mathbf{u} is the map from $S^{\mathbb{N}}$ to \mathfrak{M} defined by $\mathbf{u}((x_j)_{j=1}^{\infty}) = \sum_{j=1}^{\infty} \delta_{x_j}$.

Then we have the following theorem.

Theorem 3.3 ([19]). *Suppose that the assumptions in Theorem 3.1 are satisfied. Furthermore assume (A4)–(A6). Then, for μ -a.s. ξ , ISDE (1.6) with $\mathbf{X}(0) = \mathfrak{l}(\xi)$ has a strong solution satisfying the μ -absolute continuity condition, and that pathwise uniqueness holds for ISDE (1.6) with the μ -absolute continuity condition.*

§ 4. Applications

Theorems 3.1 and 3.3 can be applied to quite general class of ISDEs. In this section we give some important examples.

Example 4.1 (Canonical Gibbs measures). Let $S = \mathbb{R}^d$, $d \in \mathbb{N}$. Assume that $\Phi = 0$ and that Ψ_0 is a super stable and regular in the sense of Ruelle [22], and is smooth outside the origin. Let μ be a canonical Gibbs measure with the interaction Ψ_0 . Then its logarithmic derivative is

$$(4.1) \quad \mathbf{d}^\mu \left(x, \sum_{k:k \neq j} \delta_{y_k} \right) = - \sum_{k=1, k \neq j}^{\infty} \nabla \Psi_0(x - y_k).$$

Assume that (A.2) is satisfied. In the case $d \geq 2$, there exists a diffusion process associated with μ and the labeled process solves

$$(4.2) \quad dX_j(t) = dB_j(t) - \frac{1}{2} \sum_{k=1, k \neq j}^{\infty} \nabla \Psi_0(X_j(t) - X_k(t)) dt.$$

In the case $d = 1$, Ψ_0 needs to be sufficient repulsive at the origin to satisfy (A.3).

Assume that (A.5) is satisfied and that, for each $n \in \mathbb{N}$, there exist positive constants c, c' satisfying

$$(4.3) \quad \sum_{r=1}^{\infty} \frac{\int_{|x|>r} \rho^1(x) dx}{r^c} < \infty,$$

$$(4.4) \quad \sum_{i,j=1}^d \left| \frac{\partial^2}{\partial x_i \partial x_j} \Psi_0(x) \right| \leq \frac{c'}{(1+|x|)^{c'+1}},$$

for all $|x| \geq 1/n$. In [19, Theorem 3.3] it was proved that, for μ -a.s. ξ , ISDE (4.2) with $\mathbf{X}(0) = \mathbf{l}(\xi)$ has a strong solution satisfying the μ_{Tail}^ξ -absolute continuity condition, and that pathwise uniqueness holds for ISDE (1.6) with the μ_{Tail}^ξ -absolute continuity condition.

Example 4.2 (Sine random point fields). Let $\check{\mu}_{\text{bulk},\beta}^N$ be the probability measure defined in (1.3). We denote by $\mu_{\text{bulk},\beta}^N$ the distribution of $\sum_{j=1}^N \delta_{x_j}$ under $\check{\mu}_{\text{bulk},\beta}^N$. For $\beta > 0$ the existence of the limit of $\mu_{\text{bulk},\beta}^N$ as $N \rightarrow \infty$ was shown in Valkó-Virág[24]. We denote the limit by $\mu_{\text{bulk},\beta}$. In particular, when $\beta = 2$, $\mu_{\text{bulk},2}$ is the determinantal point process (DPP) with the sine kernel

$$(4.5) \quad K_{\text{sin},2}(x, y) = \frac{\sin(x - y)}{\pi(x - y)},$$

and when $\beta = 1, 4$, it is a quaternion determinantal point process [2]. It was shown that $\mu_{\text{bulk},\beta}$ for $\beta = 1, 2, 4$ is a quasi-Gibbs measure in [13], and that its logarithmic derivative is

$$(4.6) \quad \mathbf{d}^\mu \left(x, \sum_{k:k \neq j} \delta_{y_k} \right) = \beta \lim_{r \rightarrow \infty} \sum_{\substack{k:k \neq j \\ |y_k| < r}} \frac{1}{x - y_k}$$

in [12]. In [19, Theorem 3.1] it was shown that for $\mu_{\text{bulk},\beta}$ -a.s. ξ , ISDE (1.4) with $\mathbf{X}(0) = \mathbf{l}(\xi)$ has a strong solution satisfying the $\mu_{\text{bulk},\beta,\text{Tail}}^\xi$ -absolute continuity condition, and that pathwise uniqueness holds for ISDE (1.4) with the $\mu_{\text{bulk},\beta,\text{Tail}}^\xi$ -absolute continuity condition. In the case $\beta = 2$, the facts that $\mathcal{T}(\mathfrak{M})$ is $\mu_{\text{bulk},2}$ -trivial and $\mu_{\text{bulk},2,\text{Tail}}^\xi = \mu_{\text{bulk},2}$ were shown in [15].

Tsai [23] proved the existence and uniqueness of solutions of ISDE (1.4) for $\beta \geq 1$ by a different method. Thus it is conjectured that $\mu_{\text{bulk},\beta}$ is a quasi-Gibbs measure and has a logarithmic derivative of the form (4.6) for $\beta \geq 1$.

Example 4.3 (Airy random point fields). We denote by $\mu_{\text{soft},\beta}^N$ the distribution of $\sum_{j=1}^N \delta_{N^{1/6}(x_j - 2\sqrt{N})}$ under $\check{\mu}_{\text{bulk},\beta}^N$. For $\beta > 0$, the existence of the limit of $\mu_{\text{soft},\beta}^N$ as $N \rightarrow \infty$ was shown in Ramírez-Rider-Virág [21]. We denote the limit by $\mu_{\text{soft},\beta}$. In particular, when $\beta = 2$, $\mu_{\text{soft},2}$ is the DPP with the Airy kernel

$$(4.7) \quad K_{\text{Ai},2}(x, y) = \frac{\text{Ai}(x)\text{Ai}'(y) - \text{Ai}'(x)\text{Ai}(y)}{x - y},$$

where Ai denotes the Airy function and Ai' its derivative [9]. When $\beta = 1, 4$, it is a quaternion determinantal point process [2]. In the cases $\beta = 1, 2, 4$, it has been proved that the random point field is quasi-Gibbsian [14], and that its logarithmic derivative is

$$(4.8) \quad \mathbf{d}^\mu \left(x, \sum_{k:k \neq j} \delta_{y_k} \right) = \beta \lim_{r \rightarrow \infty} \left\{ \sum_{\substack{k:k \neq j \\ |y_k| < r}} \frac{1}{x - y_k} - \int_{-r}^r \frac{\widehat{\rho}(x)dx}{-x} \right\},$$

and for $\mu_{\text{soft},\beta}$ -a.s. ξ , ISDE (1.5) with $\mathbf{X}(0) = \mathbf{l}(\xi)$ has a strong solution satisfying the $\mu_{\text{soft},\beta,\text{Tail}}^\xi$ -absolute continuity condition, and pathwise uniqueness holds for ISDE (1.5) with the $\mu_{\text{soft},\beta,\text{Tail}}^\xi$ -absolute continuity condition [18, Theorem 2.3]. In the case $\beta = 2$ the facts that $\mathcal{T}(\mathfrak{M})$ is $\mu_{\text{soft},2}$ -trivial and that $\mu_{\text{soft},2,\text{Tail}}^\xi = \mu_{\text{soft},2}$ were shown in [15].

Determining whether $\mu_{\text{soft},\beta}$ has the quasi-Gibbs property for general β and finding its logarithmic derivative is (4.8) are interesting and important problems.

Example 4.4 (Bessel random point field). Let $S = [0, \infty)$ and $1 \leq \alpha < \infty$. Let $\mu_{\text{hard},2}$ be the determinantal point process with Bessel kernel

$$(4.9) \quad K_{J_\alpha}(x, y) = \frac{J_\alpha(\sqrt{x})\sqrt{y}J'_\alpha(\sqrt{y}) - \sqrt{x}J'_\alpha(\sqrt{x})J_\alpha(\sqrt{y})}{2(x - y)}.$$

In [6] it was shown that $\mu_{\text{hard},2}$ is a quasi-Gibbs measure and that the related process is the unique strong solution of the ISDE

$$dX_j(t) = dB_j(t) + \left\{ \frac{\alpha}{2X_j(t)} + \sum_{k=1, k \neq j}^{\infty} \frac{1}{X_j(t) - X_k(t)} \right\} dt$$

with the $\mu_{\text{hard},2}$ -absolute continuity condition.

Example 4.5 (Ginibre random point field). Let $S = \mathbb{R}^2$ be identified as \mathbb{C} . Let μ_{Gin} be the DPP with the kernel $K_{\text{Gin}} : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$ defined by

$$(4.10) \quad K_{\text{Gin}}(x, y) = \frac{1}{\pi} e^{-|x|^2/2 - |y|^2/2} e^{x\bar{y}}.$$

In [13] it was shown that μ_{Gin} is a quasi-Gibbs measure, and in [12] that the related process is a solution of the ISDE

$$(4.11) \quad dX_j(t) = dB_j(t) - X_j(t)dt + \lim_{r \rightarrow \infty} \sum_{\substack{k: k \neq j \\ |X_k(t)| < r}} \frac{X_j(t) - X_k(t)}{|X_j(t) - X_k(t)|^2} dt.$$

The pathwise uniqueness of solutions of (4.11) with the μ_{Gin} -absolute continuity condition was shown in [19].

§ 5. Remarks

In the previous section we gave some examples of DPPs that are not canonical Gibbs measures but quasi-Gibbs measures. It is expected that quite general DPPs have the quasi-Gibbs property. We thus present examples of DPPs related to random matrix theory or non-colliding Brownian motions, whose quasi-Gibbs property have not been shown.

Example 5.1 (Pearcey process). Consider $2N$ noncolliding Brownian motions, in which all particles start from the origin and N particles end at \sqrt{N} at time $t = 1$, and the other N particles end at $-\sqrt{N}$ at $t = 1$. We denote the system by $(X_1^N(t), \dots, X_{2N}^N(t))$, $0 \leq t \leq 1$. When N is very large, there is a cusp at $x_0^N = 0$ when $t_0 = \frac{1}{2}$, that is, before time t_0 particles are in one interval with high probability, while after time t_0 they are separated into two intervals by the origin. We denote the distribution

$$\sum_{j=1}^{2N} \delta_{2^{3/2}(2N)^{1/4} X_j^N(\frac{1}{2})}$$

on \mathfrak{M} by μ_{pearcey}^N . It was proved in Adler-Orantin-von Moerbeke [1] that

$$\mu_{\text{pearcey}}^N \rightarrow \mu_{\text{pearcey}}, \quad \text{weakly as } N \rightarrow \infty$$

and that μ_{pearcey} is the DPP $K_{\text{pearcey}}(x, y)$ given by

$$K_{\text{pearcey}}(x, y) = \frac{P(x)Q''(y) - P'(x)Q'(y) + P''(x)Q(y)}{x - y}, \quad x, y \in \mathbb{R},$$

with

$$Q(y) = \frac{i}{2\pi} \int_{-i\infty}^{i\infty} e^{-u^4/4 - uy} du \quad \text{and} \quad P(x) = \frac{1}{2\pi i} \int_C e^{v^4/4 + vx} dv,$$

where the contour C is given by the ingoing rays from $\pm\infty e^{i\pi/4}$ to 0 and the outgoing rays from 0 to $\pm\infty e^{-i\pi/4}$. These integrals are known as Pearcey's integrals [20].

Example 5.2 (Tacnode process). Consider two groups of non-colliding pinned Brownian motions $(X_1^N(t), \dots, X_{2N}^N(t))$ in the time interval $0 \leq t \leq 1$, where one group of N particles starts and ends at \sqrt{N} and the other group of N particles starts and ends at $-\sqrt{N}$. The distribution $(N^{1/6}X_1^N(\frac{1}{2}), N^{1/6}X_2^N(\frac{1}{2}), \dots, N^{1/6}X_{2N}^N(\frac{1}{2}))$ on the Weyl chamber of type A_{2N-1}

$$\mathbb{W}_{2N} = \left\{ \mathbf{x} = (x_1, x_2, \dots, x_{2N}) : x_1 < x_2 < \dots < x_{2N} \right\},$$

is given by

$$m_{\text{tac}}^{2N}(d\mathbf{x}_{2N}) = \frac{1}{Z} \left[\det_{1 \leq i, j \leq 2N} \left(e^{-2|x_i - a_j|^2} \right) \right]^2,$$

where $a_j = -\sqrt{N}$ for $1 \leq j \leq N$ and $a_j = \sqrt{N}$ for $N+1 \leq j \leq 2N$. We denote the distribution of $\sum_{j=1}^{2N} \delta_{N^{1/6}x_j}$ under m_{tac}^{2N} by μ_{tac}^N . It was proved in Delvaux-Kuijlaars-Zhang [3] and Johansson [7] that

$$\mu_{\text{tac}}^N \rightarrow \mu_{\text{tac}}, \quad \text{weakly as } N \rightarrow \infty$$

and that μ_{tac} is the DPP with the correlation kernel

$$K_{\text{tac}}(x, y) \equiv L_{\text{tac}}(x, y) + L_{\text{tac}}(-x, -y), \quad x, y \in \mathbb{R},$$

where

$$\begin{aligned} L_{\text{tac}}(x, y) &= K_{\text{Ai}, 2}(x, y) \\ &\quad + 2^{1/3} \int_{(0, \infty)^2} dudv \text{Ai}(y + 2^{1/3}u)R(u, v)\text{Ai}(x + 2^{1/3}v) \\ &\quad - 2^{1/3} \int_{(0, \infty)^2} dudv \text{Ai}(-y + 2^{1/3}u)\text{Ai}(u + v)\text{Ai}(x + 2^{1/3}v) \\ &\quad - 2^{1/3} \int_{(0, \infty)^3} dudvdw \text{Ai}(-y + 2^{1/3}u)R(u, v)\text{Ai}(v + w)\text{Ai}(x + 2^{1/3}w). \end{aligned}$$

Here, $R(x, y)$ is the resolvent operator for the restriction of the Airy kernel to $[0, \infty)$, that is, the kernel of the operator

$$(5.1) \quad R = (I - K_{\text{Ai}})^{-1} K_{\text{Ai}}$$

on $L^2[0, \infty)$.

In [3, 7] it was also shown that

$$\Xi^N(t) \equiv \sum_{j=1}^{2N} \delta_{N^{1/6} X_j(\frac{1}{2} + N^{-1/3}t)} \rightarrow \Xi(t), \quad \text{as } N \rightarrow \infty,$$

in the sense of finite-dimensional distributions, where $\Xi(t)$ is a reversible process with reversible measure μ_{tac} . We expect that $\Xi(t)$ is the diffusion process associated with the Dirichlet form $(\mathcal{E}^{\mu_{\text{tac}}}, \mathcal{D}^{\mu_{\text{tac}}})$.

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